AP® CALCULUS AB/CALCULUS BC
2014 SCORING GUIDELINES

Question 1

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

Solution	Scoring
(a) $\dfrac{A(30) - A(0)}{30 - 0} = -0.197$ (or -0.196) lbs/day	1 : answer with units
(b) $A'(15) = -0.164$ (or -0.163) The amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lbs/day at time $t = 15$ days.	$2 : \begin{cases} 1 : A'(15) \\ 1 : \text{interpretation} \end{cases}$
(c) $A(t) = \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \Rightarrow t = 12.415$ (or 12.414)	$2 : \begin{cases} 1 : \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \\ 1 : \text{answer} \end{cases}$
(d) $L(t) = A(30) + A'(30)\cdot(t - 30)$ $A'(30) = -0.055976$ $A(30) = 0.782928$ $L(t) = 0.5 \Rightarrow t = 35.054$	$4 : \begin{cases} 2 : \text{expression for } L(t) \\ 1 : L(t) = 0.5 \\ 1 : \text{answer} \end{cases}$

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

$$\frac{A(30) - A(0)}{30 - 0} = \frac{-5.904}{30} \approx -.197 \ \frac{\text{pounds}}{\text{day}}$$

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

$$A'(t) \approx -.478(.931)^t$$

$$A'(15) \approx -.164 \ \frac{\text{pounds}}{\text{day}}$$

The amount of grass clippings in the bin is decreasing (decomposing) at a rate of .164 pounds per day at time = 15 days

Continue problem 1 on page 5.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

average amount $= \frac{1}{30}\int_0^{30} A(t)\,dt \approx 2.75263511$

$A(t) = 2.75263511 = 6.687(.931)^t$

this occurs at $t \approx 12.414$ days

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

$L(t)$ is the tangent line to $A(t)$ at $t=30$

$A(30) \approx .783 \Rightarrow (30, .783) = (t, A(t))$

$A'(30) \approx -.056$ let $-.056 = M$

$(y - y_1) = M(x - x_1)$ so for this problem,

$(A(t) - .783) = -.056(t-30)$

when there are .5 pounds of grass, $A(t) = .5$,

$(.5 - .783) = -.056(t-30)$

$t = 35.054$ days

GO ON TO THE NEXT PAGE.

 1B₁

Do not write beyond this border.

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

$$\frac{A(30) - A(0)}{30 - 0} = \frac{.7829178 - 6.687}{30} = -.196 \frac{\text{pounds}}{\text{day}}$$

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

$$A'(t) = -.47809(.931)^x \Big|_{x=15} = -.163 \frac{\text{pounds}^2}{\text{day}}$$

this value represents the rate of which the grass is decomposing @ t=15 days in pound² per day

Continue problem 1 on page 5.

1B2

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

$$\frac{1}{30-0}\int_0^{30} A(t)\,dt = 2.752$$

$$2.752 = 6.687(.931)^t$$

$$t = 12.418$$

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

$$A(30) = .783$$

$$(30, .783)$$

$$A'(30) = -.47809(.931)^{30} = -.056$$

$$L(t) - .783 = -.056(t-30)$$

$$L(t) = -.056t + 1.68 + .783$$

$$L(t) = -.056t + 2.463$$

$$.5 = -.056t + 2.463$$

$$t = 35.053$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $\overset{\text{in bin}}{A(t)} = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

$a^u = \ln a \cdot a^u \cdot u'$

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

$$\frac{1}{30-0}\int_0^{30} A'(t)\,dt = \frac{1}{30}\cdot 6.687 \cdot \int_0^{30} [(\ln .931)(.931^t)]\,dt$$

$= -.1968024044$ lbs of grass/day

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

$A'(t) = 6.687 \cdot \ln(.931) \cdot (.931^t)$

$A'(15) = 6.687 \cdot \ln(.931) \cdot (.931^{15}) = -.1635905804$

the rate at which the amount of grass in the bin is decomposing is $-.1635905805$ pounds per day

Unauthorized copying or reuse of any part of this page is illegal.

Continue problem 1 on page 5.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

$$A(t) = \frac{1}{30}\int_0^{30} A(t)\,dt$$

$$6.687(.931)^t = \frac{1}{30}\left[(6.687(.931)^{30}) - (6.687(.931)^{0})\right]$$

$$6.687(.931)^t = 24981.18241$$

~~there~~ is not t

Do not write beyond this border.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

$$.5 = L(t)$$

Question 1

Overview

In this problem students were given $A(t)$, a model for the amount of grass clippings, in pounds, contained in a bin at time t days for $0 \le t \le 30$. In part (a) students were asked to show the calculation of the average rate of change of $A(t)$ over the interval $0 \le t \le 30$ and specify the units of the result – pounds per day. In part (b) students were asked to calculate the derivative of $A(t)$ at $t = 15$, either by using the calculator or by applying basic derivative formulas to $A(t)$ to obtain $A'(t)$ and then evaluating $A'(t)$ at $t = 15$. This answer is negative. Therefore, students needed to interpret the absolute value of this answer as the rate at which the amount of grass clippings in the bin is decreasing, in pounds per day, at time $t = 15$ days. In part (c) students were given two tasks. First, students needed to set up and evaluate the integral expression for the average value of $A(t)$ over the interval $0 \le t \le 30$, namely $\frac{1}{30}\int_0^{30} A(t)\,dt$. Second, students needed to set up and solve the equation $A(t) = \frac{1}{30}\int_0^{30} A(t)\,dt$ for t in the interval $0 \le t \le 30$. In part (d) students needed to compute $A(30)$, $A'(30)$, and write $L(t) = A(30) + A'(30)(t - 30)$. Students were to then solve the equation $L(t) = 0.5$.

Sample: 1A
Score: 9

The student earned all 9 points. In part (d) the student evidently stored more accurate intermediate values in the calculator because the correct answer is presented.

Sample: 1B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for $A'(15)$. The units are incorrect, so the interpretation point was not earned. The student's description seems to suggest that the student thinks that $A'(15)$ is a rate of a rate or that $A'(15)$ suggests that A is decreasing at a negative rate at $t = 15$. In part (c) the student earned the first point for $\frac{1}{30 - 0}\int_0^{30} A(t)\,dt$. The student's answer is not accurate to three decimal places. In part (d) the student earned the first two points for $L(t)$ and the point for setting $L(t) = 0.5$. The student's answer is not accurate to three decimal places.

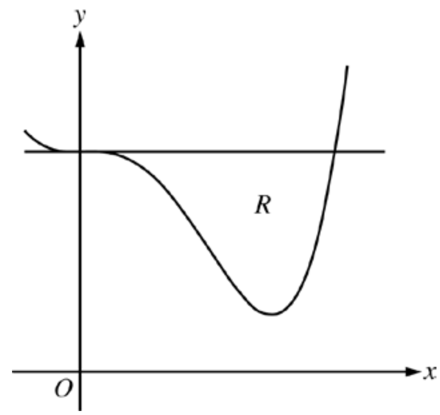
Question 1 (continued)

Sample: 1C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for $A'(15)$. The interpretation point was not earned because the student's claim that the grass clippings decompose at a negative rate is incorrect. The grass clippings are decreasing at the rate of $|A'(15)|$ pounds per day at $t = 15$. In part (c) the student earned the first point for $\frac{1}{30}\int_0^{30} A(t)\,dt$. In part (d) the student is not eligible for the point for setting $L(t) = 0.5$ because neither of the first 2 points was earned.

Question 2

Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.



(a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.

(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.

(a) $f(x) = 4 \Rightarrow x = 0, 2.3$

$$\text{Volume} = \pi\int_0^{2.3}\left[(4+2)^2 - (f(x)+2)^2\right] dx$$
$$= 98.868 \text{ (or } 98.867)$$

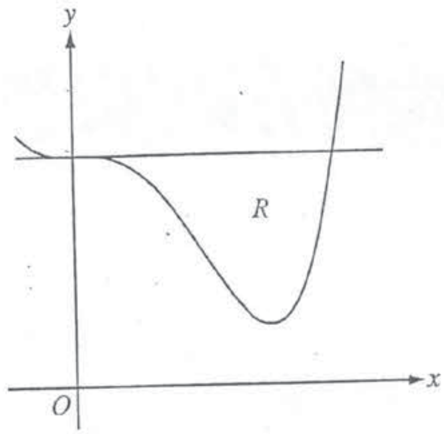
$4 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) $$\text{Volume} = \int_0^{2.3} \frac{1}{2}(4 - f(x))^2\, dx$$
$$= 3.574 \text{ (or } 3.573)$$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $$\int_0^k (4 - f(x))\, dx = \int_k^{2.3} (4 - f(x))\, dx$$

$2 : \begin{cases} 1 : \text{area of one region} \\ 1 : \text{equation} \end{cases}$



2. Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

$f(x)=4\Big|_{x=0 \text{ or } 2.3}$ $\pi\int_0^{2.3}\left((2+4)^2-(2+f(x))^2\right)dx = 98.868 \text{ units}^3$

$V = \pi(R^2 - r^2)h$

Do not write beyond this border.

Continue problem 2 on page 7.

(b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.

$A = \frac{1}{2} b h$

$A = \frac{1}{2} b^2$

$b = h$

$b = 4 - f(x)$

$$\frac{1}{2} \int_0^{2.3} (4 - f(x))^2 \, dx = 3.574 \text{ units}^3$$

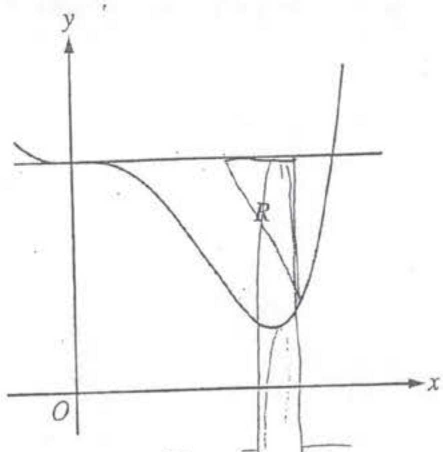
(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.

$$\int_0^k (4 - f(x)) \, dx = \frac{1}{2} \int_0^{2.3} (4 - f(x)) \, dx$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

 2B1



2. Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

$O.R. = 4 \cancel{+} -2)$

$I.R. = x^4 - 2.3x^3 + 4$

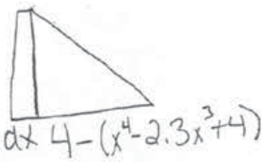
$$\pi \int_0^{2.3} [6^2 - (x^4 - 2.3x^3 + 4)^2]\,dx = 202.940$$

Do not write beyond this border.

Continue problem 2 on page 7.

2B2

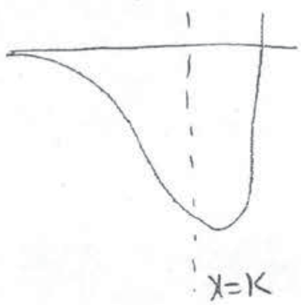
(b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.



$$A = \frac{(4-(x^4-2.3x^3+4))^2}{2}$$

$$\int_0^{2.3} \left[\frac{(4-(x^4-2.3x^3+4))^2}{2}\right] dx = 7.147$$

(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.



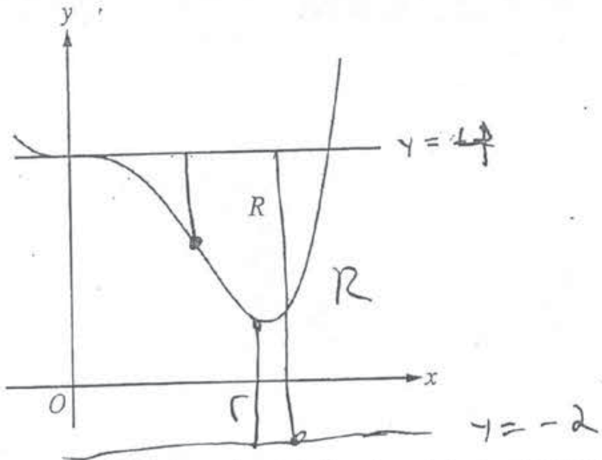
$$\int_0^{k} [4-(x^4-2.3x^3+4)]\,dx = \int_k^{2.3} [4-(x^4-2.3x^3+4)]\,dx$$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

2C1



2. Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

$\pi \int_0^{2.3} \left((4) - (x^4 - 2.3x^3 + 4) + 2\right)^2 dx$

$4 = x^4 - 2.3x^3 + 4$

$x = 2.3$

24.22012π

91.79771 units^3

Do not write beyond this border.

Continue problem 2 on page 7.

2C2

(b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.

$A = \frac{1}{2}bh$

$$\pi \int_0^{2.3} \frac{1}{2} \cdot \left(4-(x^4-2.3x^3+4)\right) \left(\frac{(4-(x^4-2.3x^3+4))}{\sqrt{2}}\right) dx$$

$$\frac{1}{2}(3.2181715)\left(\frac{3.2181715}{\sqrt{2}}\right)$$

$$11.50332 \text{ units}^3$$

Isosceles: $\sqrt{3}$? ----

(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.

$$\int_0^k \left(4-(x^4-2.3x^3+4)\right) dx + \int_k^{2.3} \left(4-(x^4-2.3x^3+4)\right) dx$$

$$=$$

$$\int_0^{2.3} \left(4-(x^4-2.3x^3+4)\right) dx$$

GO ON TO THE NEXT PAGE.

Question 2

Overview

In this problem a sketch of the boundary curves of a planar region R in the first quadrant was given. One boundary is the graph of $f(x) = x^4 - 2.3x^3 + 4,$ and the other boundary is the line $y = 4.$ In part (a) students were expected to compute the volume of the solid generated when R is rotated about the horizontal line $y = -2,$ using the method of washers. Both the integral setup and evaluation were required. Students needed to find the limits of integration and the integrand. The limits of integration are the solutions of $f(x) = 4.$ The solutions can be found by algebra or using the calculator. By the method of washers, the integrand is $\pi\left(6^2 - (f(x) + 2)^2\right)$ because the outer radius of the washer centered at $(x, 0)$ is $4 + 2 = 6$ and the inner radius of that washer is $f(x) + 2.$ Students were expected to evaluate the resulting integral by using the calculator. In part (b) students were expected to find the volume of the solid by integrating $A(x),$ the area of the cross section of the solid at $(x, 0),$ from $x = 0$ to $x = 2.3.$ By geometry, $A(x) = \left(\frac{1}{2}\right)(4 - f(x))^2.$ In part (c) students were expected to realize that the area inside R to the left of $x = k$ can be written as $\int_0^k (4 - f(x))\,dx$ and the area inside R to the right of $x = k$ can be written as $\int_k^{2.3} (4 - f(x))\,dx.$ Thus, if the vertical line $x = k$ divides R into two regions with equal areas, then $\int_0^k (4 - f(x))\,dx = \int_k^{2.3} (4 - f(x))\,dx.$

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student presents the correct square of the outer radius, but the student does not present the correct square of the inner radius. The student earned 1 of the 2 integrand points and the limits point. The student is not eligible for the answer point. In part (b) the student has a correct integrand for the cross-sectional area. The volume is not calculated correctly. In part (c) the student's work is correct.

Question 2 (continued)

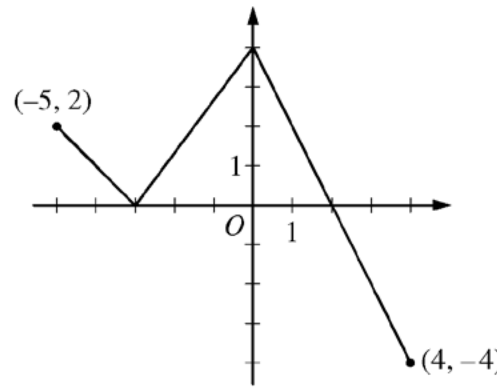
Sample: 2C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student has neither the square of the outer radius nor the square of the inner radius. The student did not earn any integrand points, but the limits point was earned. The student is not eligible for the answer point. In part (b) the student attempts to work with the area of a cross section involving an isosceles right triangle. The student presents a correct expression for the length of one of the sides of the triangle, but presents an incorrect expression for the length of the other side. The student earned 1 of the 2 integrand points and is not eligible for the answer point. In part (c) the student has the parameter k as the upper limit in an integral expression for the area of a portion of the region R. The student earned the point for the area of one region. Although the student writes an equation, the equation is true for any value of k. The student did not earn the equation point.

Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.



Graph of f

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

(a) $g(3) = \int_{-3}^{3} f(t)\,dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \dfrac{5xg'(x) - g(x)5}{(5x)^2} = \dfrac{5xg'(x) - 5g(x)}{25x^2}$

$$h'(3) = \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2}$$

$$= \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3}$$

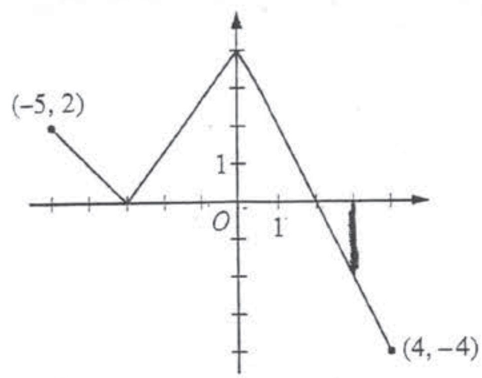
$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d) $p'(x) = f'\left(x^2 - x\right)(2x - 1)$

$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$

$3 : \begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$$g(3) = \int_{-3}^{3} f(t)\,dt$$

$$= \tfrac{1}{2}(5)(4) - \tfrac{1}{2}(1)(2)$$

$$= 10 - 1 = \boxed{9}$$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$$g'(x) > 0 \Leftrightarrow f(x) > 0$$

$$g''(x) < 0 \Leftrightarrow f'(x) < 0$$

$$(-5,-3),\ (0,2)$$

Continue problem 3 on page 15.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$h(x) = \frac{g(x)}{5x}$ $\quad h'(x) = \frac{(5x)(g'(x)) - g(x)\cdot 5}{25x^2}$

$h'(3) = \frac{(15)(f(3)) - g(3)\cdot 5}{9\cdot 25} = \boxed{\frac{(15)(-2) - (9)(5)}{9\cdot 25}}$

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$p(x) = f(x^2 - x)$

$p'(x) = f'(x^2 - x)\cdot(2x-1)$

$p'(-1) = f'(1+1)\cdot(-2-1) = f'(2)\cdot -3$

$f'(2) = \frac{-4-4}{4-0} = \frac{-8}{4} = -2$

$= (-2)(-3)$

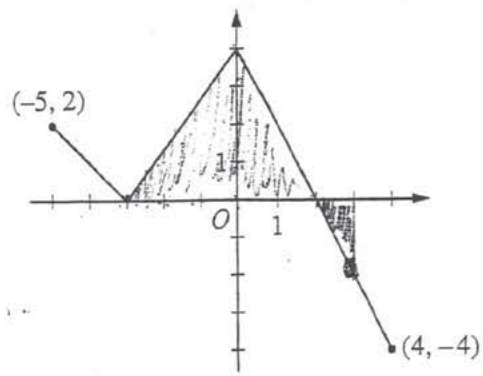
$= \boxed{6}$

Do not write beyond this border.

Do not write beyond this border.

NO CALCULATOR ALLOWED

Do not write beyond this border.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$\int_{-3}^{3} f(t)\,dt.$

$5 \cdot 4 = \frac{20}{2} = 10 + \frac{1(2)}{2}$

$= 11$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$g'(x) = f(x)$

$(0, 2)$

because g' is both positive and decreasing

when g' is positive g is incr.
when g' is dec g is concave down

Unauthorized copying or reuse of any part of this page is illegal.

Continue problem 3 on page 15.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$g'(3) = -2$

$g(3) = 11$

$$h'(x) = \frac{5x(g'(x)) - g(x) \cdot 5}{25x^2}$$

$$h'(3) = \frac{5(3)(g'(3)) - g(3) \cdot 5}{25(9)}$$

$$h'(3) = \frac{15(-2) - 11(5)}{225}$$

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$p'(x) = f'(x^2 - x) \cdot (2x - 1)$

$p'(-1) = f'(0)(-3)$

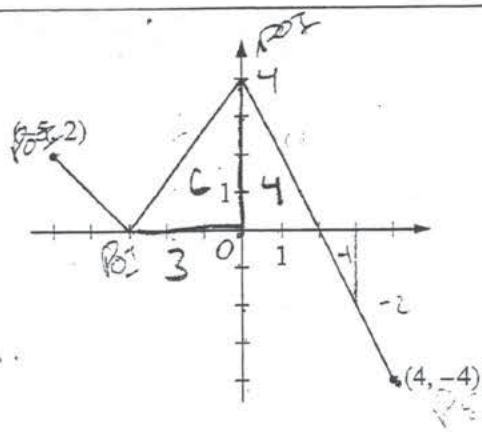
$p'(-1) = -18$

$f'(0) = g(0)$

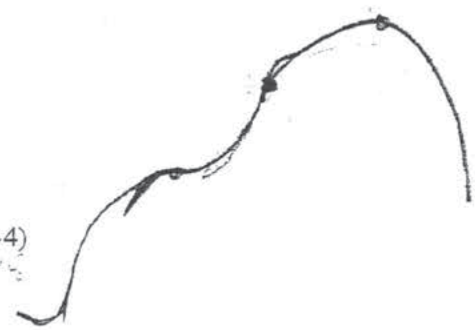
$g(0) = 6$

$f'(0) = 6$

Do not write beyond this border.



Graph of f



3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$g(3) = 10 + \int_{2}^{3} -2x+4 \, dx \qquad -x^2+4x \Big|_2^3 = (-9+12)-(-4+8)$

$= 3-4 = -1$

$g(3) = \underline{9}$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$(-5,-2)\ (0,2)$

In these x-values f(x) is positive and the slopes of f(x) show that they are concave down because they are positive

Continue problem 3 on page 15.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$$h'(x) = \frac{5x f(x) - 5g(x)}{25x^2}$$

$$h'(3) = \frac{15f(3) - 5(g(3))}{25 \cdot 9} = \frac{-30 - 40}{25 \cdot 9}$$

$$h'(3) = \boxed{\frac{-70}{225}}$$

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$$f(x) = \frac{4}{3}x + 4$$

$$f(x^2 - x) = \frac{4}{3}(x^2 - x) + 4$$

$$p(x) = \frac{4}{3}(x^2 - x) + 4$$

$$p'(x) = \frac{4}{3}(2x - 1)$$

$$m_{t_{x=-1}} = \frac{4}{3}(2(-1) - 1) = \boxed{-4}$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 3

Overview

In this problem students were given the graph of a piecewise continuous function f defined on the closed interval $[-5, 4]$. The graph of f consists of line segments whose slopes can be determined precisely. A second function g is defined by $g(x) = \int_{-3}^{x} f(t)\,dt$. In part (a) students must calculate $g(3) = \int_{-3}^{3} f(t)\,dt$ by using a decomposition of $\int_{-3}^{3} f(t)\,dt$, such as $\int_{-3}^{3} f(t)\,dt = \int_{-3}^{2} f(t)\,dt + \int_{2}^{3} f(t)\,dt$, and by applying the relationship between the definite integral of a continuous function and the area of the region between the graph of that function and the x-axis. In part (b) students were expected to apply the Fundamental Theorem of Calculus to conclude that $g'(x) = f(x)$ on the interval $[-5, 4]$. Students were to then conclude that $g''(x) = f'(x)$ wherever $f'(x)$ is defined on $[-5, 4]$. Students needed to explain that the intervals $(-5, -3)$ and $(0, 2)$ are the only open intervals where both $g'(x) = f(x)$ is positive and decreasing. In part (c) students were expected to apply the quotient rule to find $h'(3)$ using the result from part (a) and the value $g'(3) = f(3)$ from the graph of f. In part (d) students were expected to apply the chain rule to find $p'(-1)$. This required finding $f'(2)$ from the graph of f.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: no points in part (a), 1 point in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the student reports an incorrect value for $g(3)$. In part (b) the student gives an incomplete answer, but the student is eligible for and earned the reason point. In part (c) the student's work is correct based on the imported incorrect value for $g(3)$. In part (d) the student earned both derivative points but reports an incorrect value of $p'(-1)$.

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student provides values outside of the given intervals, so the student is not eligible for the reason point. In part (c) the student's derivative is correct, but the answer is incorrect. In part (d) the student presents an incorrect expression for $p(x)$ near $x = -1$, so the student is not eligible for any points.

Question 4

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

(a) average accel $= \dfrac{v_A(8) - v_A(2)}{8 - 2} = \dfrac{-120 - 100}{6} = -\dfrac{110}{3}$ m/min^2

1 : average acceleration

(b) v_A is differentiable $\Rightarrow$ v_A is continuous

$v_A(8) = -120 < -100 < 40 = v_A(5)$

Therefore, by the Intermediate Value Theorem, there is a time t, $5 < t < 8$, such that $v_A(t) = -100$.

$2 : \begin{cases} 1 : v_A(8) < -100 < v_A(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$

(c) $s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt$

$$\int_2^{12} v_A(t)\,dt \approx 3 \cdot \frac{100 + 40}{2} + 3 \cdot \frac{40 - 120}{2} + 4 \cdot \frac{-120 - 150}{2} = -450$$

$s_A(12) \approx 300 - 450 = -150$

The position of Train A at time $t = 12$ minutes is approximately 150 meters west of Origin Station.

$3 : \begin{cases} 1 : \text{position expression} \\ 1 : \text{trapezoidal sum} \\ 1 : \text{position at time } t = 12 \end{cases}$

(d) Let x be train A's position, y train B's position, and z the distance between train A and train B.

$z^2 = x^2 + y^2 \Rightarrow 2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$

$x = 300,\ y = 400 \Rightarrow z = 500$

$v_B(2) = -20 + 120 + 25 = 125$

$500\dfrac{dz}{dt} = (300)(100) + (400)(125)$

$\dfrac{dz}{dt} = \dfrac{80000}{500} = 160$ meters per minute

$3 : \begin{cases} 2 : \text{implicit differentiation of distance relationship} \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters / minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\frac{v(8) - v(2)}{8-2} \rightarrow \frac{-120 - 100}{6} \rightarrow \frac{-220}{6} \rightarrow \boxed{\frac{-110}{3} \text{ m/min}^2}$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, because $v(8) = -120$ and $v(5) = 40$ and the function is differentiable and thus continuous, the train's velocity must be -100 m/min at some point between $5 < t < 8$ according to the intermediate value theorem.

Continue problem 4 on page 17.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

$$x(12) = \int_2^{12} v_A(t)\,dt + x(2) \rightarrow \boxed{x(12) = \int_2^{12} v_A(t)\,dt + 300}$$

$$x(12) \approx 3 \cdot \tfrac{1}{2} \cdot (140) + 3 \cdot \tfrac{1}{2} \cdot (-80) + 4 \cdot \tfrac{1}{2} \cdot (-270) + 300$$

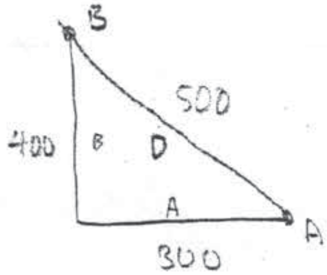
$$210 - 120 - 540 + 300$$

$$210 - 240 - 120$$

$$-30 - 120$$

$\boxed{-150}$ meaning it is 150 m W of origin station

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.



$\frac{dB}{dt} = -5t^2 + 60t + 25 \rightarrow 125$ m/min

$-20 + 120 + 25$

$120 + 5 \rightarrow 125$

$\frac{dA}{dt} = 100$ m/min

$125 \times 800 = 100000$

$$A^2 + B^2 = D^2$$

$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2D\frac{dD}{dt}$$

$$600(100) + 800(125) = 1000\frac{dD}{dt}$$

$$60000 + 100000 = 1000\frac{dD}{dt}$$

$$\frac{160{,}000}{1000} = \frac{dD}{dt} \rightarrow \boxed{\frac{dD}{dt} = 160 \text{ m/min}}$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters / minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\bar{a} = \frac{V_A(8) - V_A(2)}{8-2} = \frac{-120-100}{4} = \frac{-220}{4}$$

$$= \boxed{-55 \text{ meters/minute}^2}$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, since $V_A(t)$ is continuous and differentiable, the velocity ~~of~~ of train A must at some time t with $5<t<8$ equal -100 meters/minute because $V_A(5) = 40$ and $V_A(8) = -120$.

Continue problem 4 on page 17.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

$X_A(12) = 300 + \int_2^{12} V_A(t)\,dt$

$= 300 + 3(100+40) + 3(40+(-120)) + 4(-120-150)$

$270 \times 4 = 1080$

$= 300 + 420 - 240 - 1080$

$1080 + 240 = -1320$

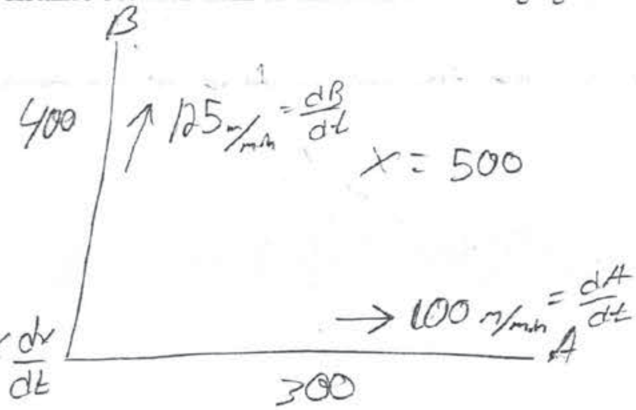
$-1320 + 720 = -600$

$\boxed{= -600 \text{ meters west of the Origin Station}}$

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

$V_B(2) = -5(2)^2 + 60(2) + 25$

$= 125$ m/min



$A^2 + B^2 = \cancel{C}x^2$

$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2x\frac{dx}{dt}$

$2(300)(100) + 2(400)(125) = 2(500)\frac{dx}{dt}$

$2(3)(100) + 2(4)(125) = 10\frac{dx}{dt}$

$600 + 1000 = 10\frac{dx}{dt}$

$\boxed{\frac{dx}{dt} = 160 \text{ m/min}}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\frac{-120-100}{8-2} = \boxed{\frac{-220}{6}} \text{ m/min}^2$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, The velocity drops from 40 m/min to -120 m/min

So at some point the velocity must have been at

-100 m/min

Do not write beyond this border.

Continue problem 4 on page 17.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

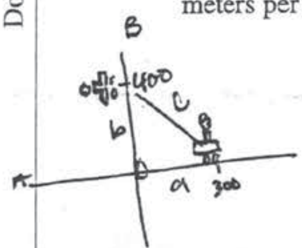
$$300 + \int_2^{12} v_A(t)\,dt \approx 300 + \frac{12-2}{6}\left(v_A(2) + 2v_A(5) + 2v_A(8) + v_A(12)\right)$$

$$= 300 + \frac{10}{6}(100 + 80 + -240 + -150)$$

$$= 300 + \frac{10}{6}(-210)$$

$$\boxed{= 300 - \frac{2100}{6}}$$ meters west of the origin Station

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.



$a^2 + b^2 = c^2$

$300^2 + 400^2 = c^2$

$\cancel{9600}$

$900 + 1600 = c^2$

$2500 = c^2$

$500 = c$

$\boxed{\text{500 meters per minute}}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 4

Overview

In this problem students were given a table of values of a differentiable function $v_A(t)$, the velocity of Train A, in meters per minute, for selected values of t in the interval $0 \le t \le 12$, where t is measured in minutes. In part (a) students were expected to know that the average acceleration of Train A over the interval $2 \le t \le 8$ is the average rate of change of $v_A(t)$ over that interval. The unit of the average acceleration is meters per minute per minute. In part (b) students were expected to state clearly that v_A is continuous because it is differentiable, and thus the Intermediate Value Theorem implies the existence of a time t between $t = 5$ and $t = 8$ at which $v_A(t) = -100$. In part (c) students were expected to show that the change in position over a time interval is given by the definite integral of the velocity over that time interval. If $s_A(t)$ is the position of Train A, in meters, at time t minutes, then $s_A(12) - s_A(2) = \int_2^{12} v_A(t)\,dt$, which implies that $s_A(12) = 300 + \int_2^{12} v_A(t)\,dt$ is the position at $t = 12$. Students approximated $\int_2^{12} v_A(t)\,dt$ using a trapezoidal approximation. In part (d) students had to determine the relationship between train A's position, train B's position, and the distance between the two trains. Students needed to put together several pieces of information from different parts of the problem and use implicit differentiation to determine the rate at which the distance between the two trains is changing at time $t = 2$.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: no points in part (a), 2 points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student makes an arithmetic mistake in computing the average acceleration. In part (b) the student's work is correct. In part (c) the student earned the point for the position expression, but the trapezoidal sum is incorrect. The student is not eligible for the answer point. In part (d) the student's work is correct.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student encloses -100 within the required interval, but the student does not provide a reason. In part (c) the position expression is correct, but the trapezoidal sum is incorrect. The student is not eligible for the answer point. In part (d) the student's work did not earn any points.

Question 5

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

(a) $x = 1$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a relative minimum at $x = 1$.

1 : answer with justification

(b) f' is differentiable $\Rightarrow$ f' is continuous on the interval $-1 \le x \le 1$

$$\frac{f'(1) - f'(-1)}{1 - (-1)} = \frac{0 - 0}{2} = 0$$

Therefore, by the Mean Value Theorem, there is at least one value c, $-1 < c < 1$, such that $f''(c) = 0$.

$2 : \begin{cases} 1 : f'(1) - f'(-1) = 0 \\ 1 : \text{explanation, using Mean Value Theorem} \end{cases}$

(c) $h'(x) = \dfrac{1}{f(x)} \cdot f'(x)$

$$h'(3) = \frac{1}{f(3)} \cdot f'(3) = \frac{1}{7} \cdot \frac{1}{2} = \frac{1}{14}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d)
$$\begin{aligned} \int_{-2}^{3} f'(g(x))g'(x)\,dx &= \Big[f(g(x)) \Big]_{x=-2}^{x=3} \\ &= f(g(3)) - f(g(-2)) \\ &= f(1) - f(-1) \\ &= 2 - 8 = -6 \end{aligned}$$

$3 : \begin{cases} 2 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

Critical numbers: $x = -1, 1$

$\xleftarrow{\quad - \quad} -1 \xrightarrow{\quad - \quad} 1 \xrightarrow{\quad + \quad}$ f'

f has a rel. min. at $x=1$ because $f'(1)=0$ and f' switches sign from negative to positive there.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

$f'(-1)=0$ and $f'(1)=0$, and $f'(x)$ is differentiable and continuous on the interval so by Rolle's Theorem there is some value c where $f''(c)=0$.

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

$$h'(x) = \frac{f'(x)}{f(x)}$$

$$h'(3) = \frac{f'(3)}{f(3)}$$

$$h'(3) = \frac{\frac{1}{2}}{7}$$

$$h'(3) = \frac{1}{2} \cdot \frac{1}{7}$$

$$\boxed{h'(3) = \frac{1}{14}}$$

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

$u = g(x)$
$du = g'(x)\,dx$

$$\int f'(u)\,du$$

$$\Big[f(g(x))\Big]_{-2}^{3}$$

$$f(g(3)) - f(g(-2))$$

$$f(1) - f(-1)$$

$$2 - 8$$

$$\boxed{-6}$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

On the interval [-2,3], the x-coordinate 0 is a relative minimum of f because f'(x) changes from negative to positive.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

There must be a value c because the Mean Value Theorem states that on a closed interval, if the function is differentiable, there must be a value c.

Continue problem 5 on page 19.

Do not write beyond this border.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

$$h'(x) = \frac{f'(x)}{f(x)}$$

$$h'(3) = \frac{f'(3)}{f(3)}$$

$$= \frac{\frac{1}{2}}{7}$$

$$\boxed{= \frac{1}{14}}$$

Do not write beyond this border.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

$$= \int_{-2}^{3} f'(g(x))g'(x)\,dx$$

$$= f(g(x))\Big|_{-2}^{3}$$

$$= f(g(3)) - f(g(-2))$$

$$= f(1) - f(-1)$$

$$= 2 - 8$$

$$\boxed{= -6}$$

NO CALCULATOR ALLOWED

inc-dec
max

dec-inc
min

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative increasing	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

-relative min when $f'(x)$ changes from decreasing to increasing
between $(-1,1)$, there is a minimum

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

Rolle's Theorem states that on an open interval $x_1 < c < x_2$, there must be a value such that $f''(c)=0$.

Do not write beyond this border.

Unauthorized copying or reuse of

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

$$h(x) = \ln(f(x))$$
$$h'(x) = \frac{1}{f(x)} \cdot f'(x)$$
$$h'(3) = \frac{1}{f(3)} \cdot f'(3)$$
$$= \frac{1}{7} \cdot \frac{1}{2}$$
$$= \frac{1}{14}$$

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$. $= f'(g(x))g'(x)\Big]_{-2}^{3}$

$$f'(g(3))g'(3) - f'(g(-2))g'(-2)$$
$$f'(1) \cdot -2 - f'(-1) \cdot 2$$
$$[\cancel{0 \cdot -2}] - [\cancel{0 \cdot 2}]$$
$$0$$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 5

Overview

In this problem students were provided with a table giving values of two twice-differentiable functions f and g at various values of x. Part (a) asked students to find the x-coordinate of each relative minimum of f on the given interval. Students should have determined that $x = 1$ is a critical point and that f' changes sign from negative to positive at that point. In part (b) students had to explain why there is a value c, for $-1 < c < 1$, such that $f''(c) = 0$. Because the function is twice differentiable, f' is continuous on the interval $-1 \le x \le 1$, and because $f'(1) = f'(-1) = 0$, the Mean Value Theorem guarantees that there is at least one value c, $-1 < c < 1$, such that $f''(c) = 0$. In part (c) students needed to differentiate $h(x)$ using the chain rule to get $h'(x) = \frac{1}{f(x)} \cdot f'(x)$. Using values from the table, $h'(3) = \frac{1}{14}$. Part (d) required students to find the antiderivative of the integrand to get $f(g(3)) - f(g(-2))$. Using values from the table, the result is -6.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: no points in part (a), no points in part (b), 3 points in part (c), and 3 points in part (d). In part (a) the student provides a seemingly correct justification but gives an incorrect answer. There is not a relative minimum at $x = 0$. In part (b) the student does not communicate that $f'(1) - f'(-1) = 0$. The student names the Mean Value Theorem but does not connect it to the question asked. The student's explanation is not complete. In parts (c) and (d), the student's work is correct.

Sample: 5C
Score: 3

The student earned 3 points: no points in part (a), no points in part (b), 3 points in part (c), and no points in part (d). In part (a) the student refers to "$f'(x)$ changes from decreasing to increasing." To earn the point, the student needs to communicate that f' changes sign from negative to positive. In part (b) the student does not communicate that $f'(1) - f'(-1) = 0$. The student names Rolle's Theorem but does not connect it to the question asked. The student's explanation is not complete. In part (c) the student's work is correct. In part (d) the student evaluates the integrand at the limits of integration without first finding an antiderivative. The student does not earn any points for use of the Fundamental Theorem of Calculus; therefore, the student is not eligible for the answer point.

Question 6

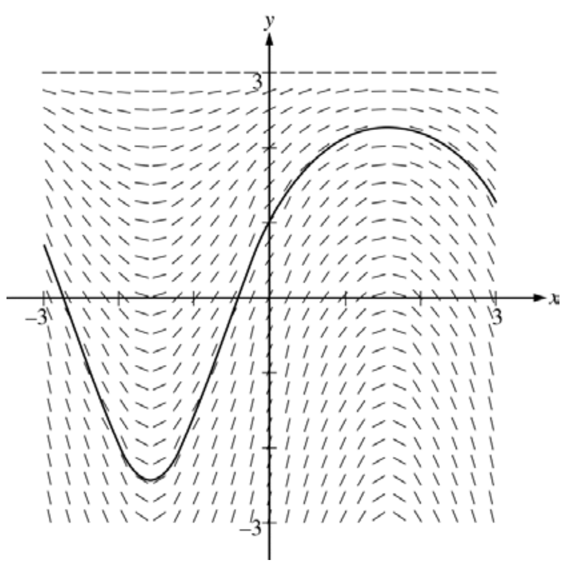
Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

(a)



1 : solution curve

(b) $\left.\frac{dy}{dx}\right|_{(x,\, y)=(0,\, 1)} = 2\cos 0 = 2$

An equation for the tangent line is $y = 2x + 1$.

$f(0.2) \approx 2(0.2) + 1 = 1.4$

$2 : \begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(c) $\frac{dy}{dx} = (3 - y)\cos x$

$\int \frac{dy}{3 - y} = \int \cos x \, dx$

$-\ln|3 - y| = \sin x + C$

$-\ln 2 = \sin 0 + C \Rightarrow C = -\ln 2$

$-\ln|3 - y| = \sin x - \ln 2$

Because $y(0) = 1$, $y < 3$, so $|3 - y| = 3 - y$

$3 - y = 2e^{-\sin x}$

$y = 3 - 2e^{-\sin x}$

Note: this solution is valid for all real numbers.

$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

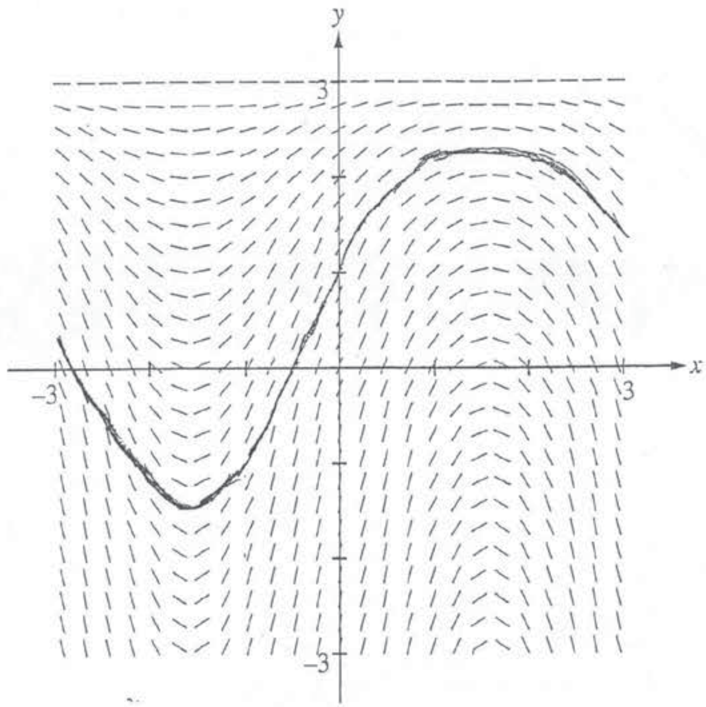
Note: $0/6$ if no separation of variables

6A₁

NO CALCULATOR ALLOWED

6. Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

$\frac{dy}{dx} = (3-y)\cos x = m$

$\frac{dy}{dx} = (3-1)\cos(0) = m$

$m = 2$

$y - 1 = 2(x - 0)$

$y - 1 = 2x$

$y = 2x + 1$

$y - 1 = 2(0.2 - 0)$

$y - 1 = 0.4$

$y = 1.4$

$\boxed{f(0.2) \approx 1.4}$

Do not write beyond this border.

Continue problem 6 on page 21.

NO CALCULATOR ALLOWED

6A₂

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

$\int \frac{dy}{3-y} = \int \cos x \, dx$

$3 - y = u$

$du = -dx$

$-\int \frac{du}{u} = \sin x + C$

$-\ln|u| = \sin x + C$

$-\ln|3-y| = \sin x + C$

$-\ln|3-1| = \sin(0) + C$

$-\ln|2| = 0 + C$

$C = -\ln 2$

$-\ln|3-y| = \sin x - \ln 2$

$\ln|3-y| = \ln 2 - \sin x$

$|3-y| = e^{\ln 2 - \sin x}$

$|3-y| = \frac{e^{\ln 2}}{e^{\sin x}}$

$|3-y| = 2e^{-\sin x}$

$3 - y = 2e^{-\sin x}$

$-y = 2e^{-\sin x} - 3$

$\boxed{y = -2e^{-\sin x} + 3}$

Do not write beyond this border.

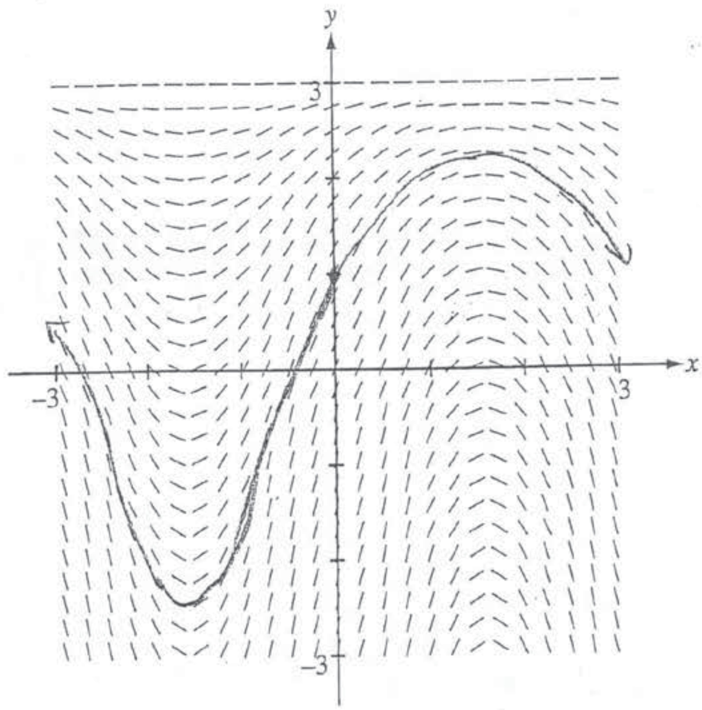
Unauthorized copying or reuse of any part of this page is illegal.

GO ON TO THE NEXT PAGE.

6B1

6. Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

$y - 1 = 2x$ - tang line

$y - y_1 = 2(x - x_1)$

$y - 1 = 2(.2 - 0)$

$y - 1 = 2(.2)$

$y - 1 = .4$

$\boxed{y = 1.4}$

Do not write beyond this border.

Continue problem 6 on page 21.

6B2

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

$$\frac{dy}{dx} = (3-y)\cos x$$

$$\int \frac{1}{3-y}\,dy = \int \cos x\,dx$$

$$\ln|3-y| = \sin x + C$$

$$\ln|3-1| = \sin(0) + C$$

$$\ln|2| = C$$

$$\ln|3-y| = \sin x + \ln|2|$$

$$3-y = \pm e^{\sin x + \ln(2)}$$

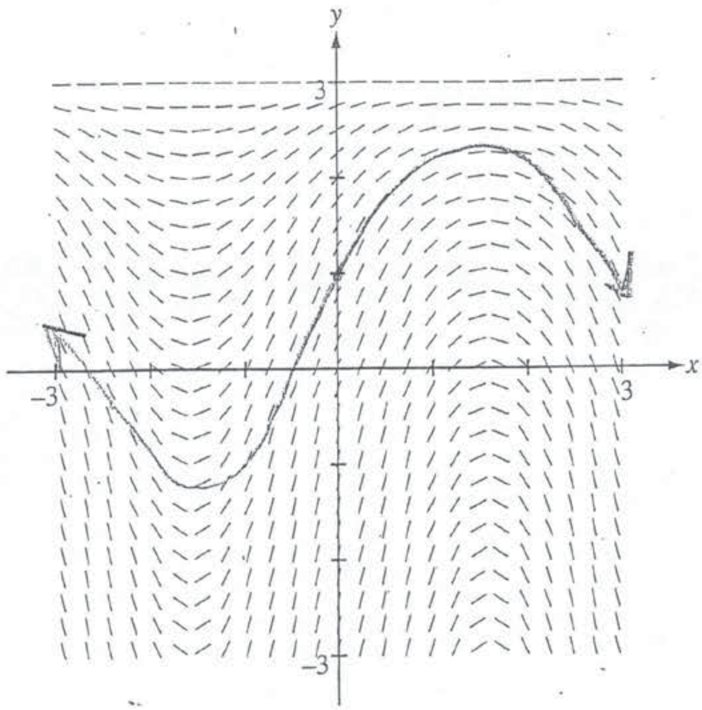
$$\boxed{y = e^{\sin x + \ln(2)} - 3}$$

Do not write beyond this border.

6C1

6. Consider the differential equation $\frac{dy}{dx} = (3-y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

$\frac{dy}{dx} = (3-1)\cos(0)$

$\boxed{\frac{dy}{dx} = 2}$

$y - 1 = 2(x - 0)$

$f(x) = 2x + 1$

$f(0.2) = 0.4 + 1$

$\boxed{f(0.2) = 1.4}$

$f(0.2)$ is 1.4

6C2

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

$$(3-y)\,dy = \cos x\,dx$$

$$3y - \frac{y^2}{2} = -\sin(x) + C$$

$$6y - y^2 = -2\sin(x) + C$$

$$y^2 - 6y = 2\sin(x) + C$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 6

Overview

This problem presented students with a differential equation and defined $y = f(x)$ to be the particular solution to the differential equation passing through a given point. Part (a) presented students with a portion of the slope field of the differential equation and asked students to draw a solution curve through the point $(0, 1)$. Part (b) asked students to write an equation for the line tangent to the solution curve from part (a) at a given point, and then to use this tangent line to approximate $f(x)$ at a nearby value of x. Students needed to recognize that the slope of the tangent line is the value of the derivative given in the differential equation at the given point. Part (c) asked for the particular solution to the differential equation that passes through the given point. Students were expected to use the method of separation of variables to solve the differential equation.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), and 4 points in part (c). In part (a) the student's solution curve is correct. In part (b) the student does not show use of $\frac{dy}{dx}$ to find an equation of the tangent line, so the first point was not earned. The student uses a correct tangent line to approximate $f(0.2)$. In part (c) the student earned the separation of variables point, 1 antiderivative point, the constant of integration point, and the initial condition point. The student has an incorrect antiderivative on the left side of the equation, so the student is not eligible for the answer point.

Sample: 6C
Score: 3

The student earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the student's solution curve is correct. In part (b) the student's work is correct. In part (c) the student's work did not earn any points.